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ASYMPTOTIC ANALYSIS OF SINGULARLY PERTURBED HAMILTON–JACOBI EQUATIONS†

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Unlike the previous investigation of the sufficient conditions for the convergence of minimax solutions of singularly perturbed Hamilton–Jacobi (H–J) equations, a typical example of which would be the Bellman–Isaacs (B–I) equations, convergence conditions are formulated not in terms of auxiliary constructs [1], but in terms of the Hamiltonian, the boundary function, assumptions regarding their continuity, Lipschitz continuity, etc. In addition, an asymptotic equation is derived, that is, a H–J equation whose minimax solution is the limit of solutions of H–J equations in which some of the momentum variables have coefficients whose denominators contain a small parameter which is made to approach zero. © 1999 Elsevier Science Ltd. All rights reserved.

The area of application of investigations of singularly perturbed H–J and B–I equations comprises problems in optimal control theory and differential games, whose dynamics involve both "fast" and "slow" motions, as well as problems with Lipschitz-continuous controls, where the Lipschitz constants may be as large as desired (see [4–14], as well as the references listed there). In such problems, the value function of a singularly perturbed B–I equation is a minimax (and/or viscosity [9–11]) solution, and problems of the existence and construction of asymptotic solutions reduce to the existence of the limit of the value functions and the corresponding unperturbed problem when the velocity of the "fast" motions or the Lipschitz constants tend to infinity.

1. FORMULATION OF THE PROBLEM. SUFFICIENT CONDITIONS FOR CONVERGENCE

Consider the following Cauchy problem $\mathbf{P}^\varepsilon$ for a singularly perturbed H–J equation ($\varepsilon \in (0, \varepsilon_*)$ is a small parameter)

$$\partial u^\varepsilon(t,x,y)/\partial t + H^\varepsilon(t,x,y,D_x u^\varepsilon, D_y u^\varepsilon) = 0 \tag{1.1}$$

$$(t,x,y) \in G^0 = (0,\theta)\times R^n \times R^l$$

$$u^\varepsilon(\theta,x,y) = \sigma(x), \quad x \in R^n, y \in R^l \tag{1.2}$$

It is assumed that the components of the vector $D_y U^\varepsilon$—the momentum variables—appear in the expression for the Hamiltonian H^ε with coefficients which contain the small parameter ε in the denominator.

As to the degree of smoothness of the initial data of problem $\mathbf{P}^\varepsilon$, it will be assumed that:

B.1. The function $\sigma(x)$ is continuous in R^n.

B.2. The Hamiltonian $H^\varepsilon(t,x,y,p,q)$ is continuous in its domain of definition $G = [0,\theta]\times R^n \times R^l \times R^n \times R^l$ and satisfies the estimate

$$\sup_{(t,x,y)\in G} \frac{|H^\varepsilon(t,x,y,0,0)|}{(1+\|x\|+\|y\|)} < \infty \tag{1.3}$$

B.3. The following Lipschitz condition holds with respect to the variables p and q, for any $(t,x,y) \in G$, $p', p'' \in R^n$, $q', q'' \in R^l$

$$|H^\varepsilon(t,x,y,p',q') - H^\varepsilon(t,x,y,p'',q'')| \leq \lambda^\varepsilon(x,y)(\|p'-p''\| + 1/\varepsilon\|q'-q''\|) \tag{1.4}$$

where $\lambda^\varepsilon(x,y) := (1+\|x\|+\|y\|)\mu^\varepsilon$ and μ^ε is a constant.

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B.4. A local Lipschitz condition holds with respect to the variables x and y

$$\frac{|H^\varepsilon(t,x',y',p,q)-H^\varepsilon(t,x'',y'',p,q)|}{(\|x'-x''\|+\|y'-y''\|)((1+\|p\|)+1/\varepsilon(1+\|q\|))}<L^\varepsilon \tag{1.5}$$

for $t\in[0,\theta], p\in R^n, q\in R^l, x', x''\in B_x, x'\neq x'', y', y''\in B_y, y'\neq y''$, where $B_x\in R^n, B_y\in R^l$ are arbitrary bounded domains, $L^\varepsilon = L^\varepsilon(B_x, B_y) = \text{const}\in(0,\infty)$.

We know [8–11] that problem $\mathbf{P}^\varepsilon$ (1.1), (1.2) does not, as a rule, have a classical solution, but conditions B.1–B.4 guarantee the existence and uniqueness, for every $\varepsilon>0$, of a generalized minimax (and/or viscosity) solution $\mu^\varepsilon(t,x,y)$ [9–11].

We recall one of several equivalent definitions of a minimax solution, which will be used in our later constructions.

Let S^ε be some non-empty set and let M^ε be a multi-valued mapping

$$G\times S^\varepsilon\ni(t,x,y,s')\mapsto M^\varepsilon(t,x,y,s')\subset R^n\times R^l\times R \tag{1.6}$$

The pair $(S^\varepsilon, M^\varepsilon)$ will be called a characteristic ε-complex of Eq. (1.1) (or, briefly, a complex) if the following conditions are satisfied.

1. For any $(t,x,y)\in G$ and $s'\in S^\varepsilon$, the set $M^\varepsilon(t,x,y,s')\subset R^n\times R^l\times R$ is non-empty, convex and closed. For any $(t,x,y,s')\in G\times S^\varepsilon$ and $(f,g,r)\in M^\varepsilon(t,x,y,s')$, one has the estimates

$$\|f\|\leq\lambda^\varepsilon(x,y),\quad \|g\|\leq\lambda^\varepsilon(x,y)$$
$$|r|\leq m^\varepsilon(t,s')(1+\|x\|+\|y\|)$$

where the quantity $\lambda^\varepsilon(x,y)$ is as defined in condition B.3. For any $s'\in S^\varepsilon$, the function $t\mapsto m^\varepsilon(t,s')$ is summable over $[0,\theta]$ and the multi-valued mapping $(t,x,y)\mapsto M^\varepsilon(t,x,y,s')$ is upper semi-continuous.

2. For any $(t,x,y)\in G$ and $p\in R^n$, $q\in R^l$

$$\text{(a)}\max_{s'\in S^\varepsilon}\min\{\langle f,p\rangle+(1/\varepsilon)\langle g,q\rangle-r:(f,g,r)\in M^\varepsilon(t,x,y,s')\}=$$
$$=H^\varepsilon(t,x,y,p,q)$$
$$\text{(b)}\min_{s'\in S^\varepsilon}\max\{\langle f,p\rangle+(1/\varepsilon)\langle g,q\rangle]-r:(f,g,r)\in M^\varepsilon(t,x,y,s')\}=$$
$$=H^\varepsilon(t,x,y,p,q)$$

The set of all complexes $(S^\varepsilon, M^\varepsilon)$ will be denoted by $\mathbf{C}(H^\varepsilon)$. We note that the above conditions hold, for example, for the pair $(S^\varepsilon, M^\varepsilon)$ with $S^\varepsilon = R^n\times R^l$, $s'=(s_1,s_2)\in S^\varepsilon$ and

$$M^\varepsilon(t,x,y,s_1,s_2)=\{(f,g,r)\in R^n\times R^l\times R:\ \|f\|\leq\lambda^\varepsilon(x,y),$$
$$\|g\|\leq\lambda^\varepsilon(x,y),\ r=\langle f,s_1\rangle+(1/\varepsilon)\langle g,s_2\rangle-H^\varepsilon(t,x,y,s_1,s_2)\}$$

where $\lambda^\varepsilon(x,y)=(1+\|x\|+\|y\|)\mu^\varepsilon$ is as defined in the Lipschitz condition B.3, $(t,x,y)\in G, s_1\in R^n$, $s_2\in R^l$.

A pair $(S^\varepsilon, M^\varepsilon)$ will be called and upper (lower) characteristic ε-complex of Eq. (1.1) if conditions 1 and 2a (conditions 1 and 2b) hold. The set of all upper (lower) characteristic ε-complexes will be denoted by $\mathbf{C}^\uparrow(H^\varepsilon)$ ($\mathbf{C}^\downarrow(H^\varepsilon)$).

Choose an arbitrary complex $(S^\varepsilon, M^\varepsilon)\in\mathbf{C}(H^\varepsilon)$ and $s'\in S^\varepsilon$. The symbol $\varepsilon-\text{Sol}(t_0,x_0,y_0,z_0,s')$ will denote the set of absolutely continuous functions $(x(\cdot),y(\cdot),z(\cdot))$: $[0,\theta]\mapsto R^n\times R^l\times R$ that satisfy the condition $(x(t_0),y(t_0),z(t_0))=(x_0,y_0,z_0)$ and the differential inclusion

$$(\dot{x}(t),\varepsilon\dot{y}(t),\dot{z}(t))\in M^\varepsilon(t,x(t),y(t),s') \tag{1.7}$$

Definition 1. An upper (lower) solution of the H–J equation (1.1) is a lower (upper) semi-continuous function $G\ni(t,x,y)\mapsto u^\varepsilon(t,x,y)\in R$ satisfying the following condition: for any $(t_0,x_0,y_0,z_0)\in\text{epi}\,u^\varepsilon$ ($(t_0,x_0,y_0,z_0)\in\text{hypo}\,u^\varepsilon$), $s'\in S^\varepsilon$ and $\tau\in[t_0,\theta]$ a trajectory $(x(\cdot),y(\cdot),z(\cdot))\in\varepsilon-\text{Sol}(t_0,x_0,y_0,z_0,s')$ exists such that $(\tau,x(\tau),y(\tau),z(\tau))\in\text{epi}\,u^\varepsilon$ ($(\tau,x(\tau),y(\tau),z(\tau))\in\text{hypo}\,u^\varepsilon$).

Here $(S^\varepsilon, M^\varepsilon)\in C^\uparrow(H^\varepsilon)$ ($(S^\varepsilon, M^\varepsilon)\in C^\downarrow(H^\varepsilon)$), $\varepsilon-\text{Sol}(t_0,x_0,y_0,z_0,s')$ is the set of trajectories of the differential inclusion (1.7) that satisfy the condition $(x(t_0),y(t_0),z(t_0))=(x_0,y_0,x_0)$.

The symbols epi u^ε and hypo u^ε denote the sets

$$\{(t,x,y,z):(t,x,y)\in G, z\geq u^{\varepsilon}(t,x,y)\}$$

$$\{(t,x,y,z):(t,x,y)\in G, z\leq u^{\varepsilon}(t,x,y)\}$$

that is, the epigraph and hypograph, respectively, of the function u^{ε}. The definition of an upper (lower) solution does not depend on the choice of the complex $(S^{\varepsilon}, M^{\varepsilon})\in C^{\uparrow}(H^{\varepsilon})$ of the complex $(S^{\varepsilon}, M^{\varepsilon})\in C^{\downarrow}(H^{\varepsilon})$).

Definition 2. A minimax solution of Eq. (1.1) is a continuous function $G\ni(t,x,y)\mapsto u^{\varepsilon}(t,x,y)\in R$ which is simultaneously an upper and a lower solution.

In what follows we will assume that:

B.5. The Hamiltonian $H^{\varepsilon}(t, x, y, p, 0)$ and conditions B.1–B.4, as well as conditions A.1 and A.2 introduced below, depend continuously on the parameter ε.

A corollary of this assumption is that the minimax solutions $u^{\varepsilon}(t, x, y)$ of problem $\mathbf{P}^{\varepsilon}$ depend continuously on the parameter $\varepsilon\in(0, \varepsilon_*]$.

To guarantee the existence of a limit of $u^{\varepsilon}(t,x,y)$ as $\varepsilon\downarrow 0$, we require the following structural conditions to hold.

A.1. Suppose characteristic complexes exist which depend continuously on the parameter $\varepsilon\in(0, \varepsilon_*]$, say $(S^{\varepsilon}_{\pm}, M^{\varepsilon}_{\pm})\in C^{\uparrow}(H^{\varepsilon})$, $(S^{\varepsilon}_{-}, M^{\varepsilon}_{-})\in C^{\downarrow}(H^{\varepsilon})$, and corresponding sets of attraction $Y^{\varepsilon}_{\pm}=Y^{\varepsilon}_{\pm}(t, x, s^{\varepsilon}_{\pm})\subset R^{l}$ with the following properties.

(a) the sets $S^{\varepsilon}_{\pm}$ do not depend on the parameter ε, and for any $S_{\pm}\in S_{\pm}$, the multi-valued mappings $(t,x,y)\mapsto M^{\varepsilon}_{\pm}(t,x,y,s_{\pm})$ are locally Lipschitz-continuous in the Hausdorff metric, with Lipschitz constants L^{ε} satisfying condition B.4

(b) for any $(t,x)\in[0,\theta]\times R^{n}$, $s_{\pm}\in S_{\pm}$, the sets $Y^{\varepsilon}_{\pm}(t,x,s_{\pm})$ are closed and bounded, and moreover

$$\forall y\in Y^{\varepsilon}_{\pm}(t,x,s_{\pm}):\ \|y\|\leq\chi^{\varepsilon}(1+\|x\|)$$
$$\chi^{\varepsilon}=\text{const},\ \chi^{\varepsilon}\in(0,\mu^{\varepsilon}]; \tag{1.8}$$

(c) for any $(t',x')\in[0,\theta]\times R^{n}$, $(t'',x'')\in[0,\theta)\times R^{n}$, $s_{\pm}\in S_{\pm}$, the following Lipschitz conditions hold

$$\operatorname{dist}(Y^{\varepsilon}_{\pm}(t',x',s_{\pm}),\ Y^{\varepsilon}_{\pm}(t'',x'',s_{\pm}))\leq\nu^{\varepsilon}(|t'-t''|+\|x'-x''\|) \tag{1.9}$$
$$\nu^{\varepsilon}=\text{const},\ \nu^{\varepsilon}\in(0,L^{\varepsilon}]$$

where $\operatorname{dist}(Y^1, Y^2)$ denotes the Hausdorff distance between the sets Y^1 and Y^2 in a finite-dimensional space

(d) for any compact sets $D\subset[0,\theta]\times R^{n}$ and $D_0\subset\mathbf{R}^{l}$

$$D_0\supset\bigcup_{(t_0,x_0)\in D, s_+\in S_+}Y^{\varepsilon}_{+}(t_0,x_0,s_+)$$

$$\left(D_0\supset\bigcup_{(t_0,x_0)\in D, s_-\in S_-}Y^{\varepsilon}_{-}(t_0,x_0,s_-)\right)$$

$\delta(\varepsilon)>$ exists such that $\delta(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$, and for any $(t_0, x_0, y_0)\in D\times D_0$ the following relationships hold

$$\begin{aligned}&\mathit{dist}(y^{\varepsilon}(t), Y^{\varepsilon}(t,x^{\varepsilon}(t),s'))\leq\operatorname{diam}D_0\quad\text{for } t\geq t_0\\&y^{\varepsilon}(t)\in Y^{\varepsilon}(t,x^{\varepsilon}(t),s'))\quad\text{for } t\in[t_0+\delta(\varepsilon),\theta]\end{aligned}\tag{1.10}$$

where $Y^{\varepsilon}=Y^{\varepsilon}_{+}$, $(Y^{\varepsilon}=Y^{\varepsilon}_{-})$.

A.2. The quantities

$$\begin{aligned}H^{\varepsilon}_{\pm}(t,x,s)=\max_{s_{\pm}\in S_{\pm}}\min\{\langle f,s\rangle-r:\\(f,r)\in\operatorname{co}\operatorname{pr}_{x,z}M^{\varepsilon}_{\pm}(t,x,Y^{\varepsilon}_{\pm}(t,x,s_{\pm}),s_{\pm})\}\end{aligned}\tag{1.11}$$

where $\mathrm{pr}_{x,z}M$ is the projection of the set M from (x, y, z)-space onto (x, z)-space, and co$Q$ denotes the convex hull of the set Q, satisfy the following inequality for any $(t, x, s) \in [0, \theta] \times R^n \times R^n$, $\varepsilon \in (0, \varepsilon_*]$

$$|H_+^\varepsilon(t,x,s) - H_-^\varepsilon(t,x,s)| \leq \alpha(\varepsilon) \tag{1.12}$$

where $\alpha(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. The limit

$$H(t,x,s) = \lim_{\varepsilon \downarrow 0} H_+^\varepsilon(t,x,s) = \lim_{\varepsilon \downarrow 0} H_-^\varepsilon(t,x,s) \tag{1.13}$$

will be regarded as the Hamiltonian in the following unperturbed Cauchy problem **P**

$$\begin{aligned} &\partial u(t,x)/\partial t + H(t,x,D_x u) = 0, \quad (t,x) \in (0,\theta) \times R^n \\ &u(\theta,x) = \sigma(x), \quad x \in R^n \end{aligned} \tag{1.14}$$

The main result of this paper is the following.

Theorem 1. Assume that conditions A.1, A.2, B.1–B.5 hold in the Cauchy problems $\mathbf{P}^\varepsilon$, $\varepsilon \in (0, \varepsilon_*]$, for the singularly perturbed H–J equation (1.1). Then the minimax solutions $u^\varepsilon(t,x,y)$ of these problems converge as $\varepsilon \downarrow 0$, $(t, x, y) \in G$ to a minimax solution $u(t, x)$ of the unperturbed problem **P** uniformly in any compact sets $D \subset [0, \theta] \times R^n$, $D_0 \subset R^l$.

Condition A.1 is not particularly easy to verify. In what follows we will assume a more convenient condition:

A.1*. Suppose that for any $(t,x,y,p,q) \in G \times R^n \times R^l$

(a) the following representation holds

$$H^\varepsilon(t,x,y,p,q) = H^\varepsilon(t,x,y,p,0) + (1/\varepsilon)h^\varepsilon(t,x,y,q)$$

$$\forall \lambda \geq 0: \ h^\varepsilon(t,x,y,\lambda q) = \lambda h^\varepsilon(t,x,y,q)$$

$$h^\varepsilon(t,x,y,q) = \langle q, k^\varepsilon(t,x,y)\rangle + \eta^\varepsilon(t,x,q)$$

where, by B.3

$$|\eta^\varepsilon(t,x,q)| \leq \tfrac{1}{2}\mu^\varepsilon(1+\|x\|)\|q\|$$

$$\|k^\varepsilon(t,x,y)\| \leq \tfrac{1}{2}\mu^\varepsilon(1+\|x\|+\|y\|)$$

($\mu^\varepsilon > 0$ is the constant from condition B.3);

(b) define sets

$$F_+^\varepsilon(t,x,q) = \{g \in R^l : \|g\| \leq \tfrac{1}{2}\mu^\varepsilon(1+\|x\|), \langle q,g\rangle \geq \eta^\varepsilon(t,x,q)\}$$

$$F_-^\varepsilon(t,x,q) = \{g \in R^l : \|g\| \leq \tfrac{1}{2}\mu^\varepsilon(1+\|x\|), \langle q,g\rangle \leq \eta^\varepsilon(t,x,q)\}$$

$$Y_\pm^\varepsilon(t,x,q) \subset \{\forall y^0 : \ -k^\varepsilon(t,x,y^0) \in F_\pm^\varepsilon(t,x,q)\}$$

and suppose that for any $(t, x) \in [0, \theta] \times R^n$, $s_\pm \in S_\pm$ the sets $Y_\pm^\varepsilon(t, x, s_\pm)$ are closed and bounded, and moreover

$$\forall y \in Y_\pm^\varepsilon(t,x,s_\pm): \ \|y\| \leq \chi^\varepsilon(1+\|x\|)$$

$$\chi^\varepsilon = \text{const}, \quad \chi^\varepsilon \in (0,\mu^\varepsilon]$$

(c) for any $(t', x') \in [0, \theta] \times R^n$, $s_\pm \in S_\pm$, the following Lipschitz conditions hold

$$\text{dist}\,(Y_\pm^\varepsilon(t',x',s_\pm), \ Y_\pm^\varepsilon(t'',x'',s_\pm)) \leq \nu^\varepsilon(|t'-t''|+\|x'-x''\|)$$

$$\nu^\varepsilon = \text{const}, \quad \nu^\varepsilon \in (0,L^\varepsilon]$$

(d) assume that for any compact set $D \subset [0, \theta] \times R^n$ a continuous mapping $D \ni (t, x) \to K^\varepsilon(t, x) \geq K^\varepsilon(D) > 0$ exists such that

$$\forall y \notin Y^{\varepsilon}_{\pm}(t,x,q)\ \exists y^{*} \in Y^{\varepsilon}_{\pm}(t,x,q):$$
$$\big\langle (y-y^{*}), (k^{\varepsilon}(t,x,y)-k^{\varepsilon}(t,x,y^{*}))\big\rangle \le -K^{\varepsilon}(t,x)\| y-y^{*}\|^{2}$$
$$\max_{g\in F^{\varepsilon}_{\pm}(t,x,q)} \big\langle (y-y^{*}), (k^{\varepsilon}(t,x,y^{*})+g)\big\rangle = -h^{\varepsilon}(t,x,y^{*},(y^{*}-y)) = 0$$

When condition A.1 is replaced by A.1*, the assertion of Theorem 1 remains valid, namely the following theorem holds.

Theorem 2. Under the assumptions A.1*, A.2, B.1–B.5, the minimax solutions $u^{\varepsilon}(t,x,y)$ of problem $\mathbf{P}^{q\varepsilon}$ converge locally uniformly as $\varepsilon \downarrow 0$ to a minimax solution $u(t,x)$ of problem **P** for all $(t,x,y) \in G$.

2. PROOF OF THEOREM 1

We will show that conditions A.1 and A.2 are analogous to conditions 3 and 4 in [1] and guarantee the convergence of $u^{\varepsilon}(t,x,y)$ to $u(t,x)$. For clarity, all further arguments will be carried out for upper characteristic complexes and the corresponding sets of attraction occurring in condition A.1. The analogous constructions and propositions for lower characteristic complexes are obtained by formally replacing the subscript "plus" by "minus".

Suppose that $(t_0, x_0, y_0) \in D \times D_0$, $z_0 \in R^1$, $\varepsilon \in (0, \varepsilon_*]$ and, in accordance with A.1, for $s_+ \in S_+$, let $Y^{\varepsilon}{}_{+}(t,x,s_+)$ be sets of attraction relative to an ε-characteristic complex $(S_+, M^{\varepsilon}{}_{+})$. Let $(x^{\varepsilon}(\cdot), y^{\varepsilon}(\cdot), z^{\varepsilon}(\cdot)) \in \varepsilon - \mathrm{Sol}(t_0, x_0, y_0, z_0, s_+)$, $s_+ \in S_+$, that is

$$(\dot{x}^{\varepsilon}(t), \varepsilon\dot{y}^{\varepsilon}(t), \dot{z}^{\varepsilon}(t)) \in M^{\varepsilon}_{+}(t, x^{\varepsilon}(t), y^{\varepsilon}(t), s_+) \tag{2.1}$$
$$(x^{\varepsilon}(t_0), y^{\varepsilon}(t_0), z^{\varepsilon}(t_0)) = (x_0, y_0, z_0)$$

In what follows we will need the following fact from the theory of differential inclusions (see [12]).

Let $(t,x,z) \mapsto F_i(t,x,z) \subset R^n \times R$: $[t_0, \theta] \times R^n \times R \mapsto 2^{R^n \times R}$ $(i = 1, 2)$ be two multi-valued mappings with convex, compact, non-empty values, which are upper semi-continuous with respect to inclusion. Let $x_0 \in R^n$, $z_0 \in R$. Consider the differential inclusions

$$(\dot{x}_i(t),\ \dot{z}_i(t)) \in F_i(t, x_i(t), z_i(t)),\quad t \in [t_0, \theta] \tag{2.2}$$
$$(x_i(t_0),\ z_i(t_0)) = (x_0, z_0),\quad i = 1, 2$$

Denote the set of solutions $(x_i(\cdot), z_i(\cdot))$ of the ith differential inclusion (2.2) by $\mathrm{Sol}_i(t_0, x_0, z_0)$. Then (see [12]) the following proposition holds.

Proposition 1. For any solution $(x_i(\cdot), z_i(\cdot)) \in \mathrm{Sol}_1(t_0, x_0, z_0)$, a solution $(x_2(\cdot), z_2(\cdot)) \in \mathrm{Sol}_2\,(t_0, x_0, z_0)$ exists such that, for all $t \in [t_0, \theta]$

$$\| w_1(t) - w_2(t) \| \le \int_{t_0}^{t} \mathrm{dist}(F_1(\tau, x_1(\tau), z_1(\tau))\ \ F_2(\tau, x_2(\tau), z_2(\tau))d\tau,\quad w = x, z \tag{2.3}$$

Thus, let us fix some solution $(x^{\varepsilon}(\cdot), y^{\varepsilon}(\cdot), z^{\varepsilon}(\cdot)) \in \varepsilon - \mathrm{Sol}\,(t_0, x_0, z_0, s_+)$, $s_+ \in S_+$ and, using the function $y^{\varepsilon}(\cdot)$: $[t_0, \theta] \mapsto D_0$, construct a multi-valued mapping

$$(t,x) \mapsto Y^{\varepsilon}_{0}(t,x,s_+) \subset Y^{\varepsilon}_{+}(t,x,s_+) \tag{2.4}$$
$$Y^{\varepsilon}_{0}(t,x,s_+) = \{y_0 \in Y^{\varepsilon}_{+}(t,x,s_+) : \mathrm{dist}(y^{\varepsilon}(t), Y^{\varepsilon}_{+}(t,x,s_+)) = \| y^{\varepsilon}(t) - y_0 \|\}$$

It can be shown that for any s_+ the mapping $(t,x) \mapsto Y^{\varepsilon}_{0}(t,x,s_+)$ is compact-valued and upper semi-continuous with respect to inclusion. But this means that the same properties will hold for the multi-valued mapping

$$(t,x) \mapsto \mathrm{co}\,\mathrm{pr}_{x,z}\, M^{\varepsilon}_{+}(t, x, Y^{\varepsilon}_{0}(t,x,s_+), s_+) \tag{2.5}$$

Now consider the differential inclusion generated by (2.5)

$$(\dot{x}_0^\varepsilon(t), \dot{z}_0^\varepsilon(t)) \in \operatorname{co}\operatorname{pr}_{x,z} M_+^\varepsilon(t, x_0^\varepsilon(t), Y_0^\varepsilon(t, x_0^\varepsilon(t), s_+), \tag{2.6}$$
$$x_0^\varepsilon(t_0) = x_0, \quad z_0^\varepsilon(t_0) = z_0$$

By the theory of differential inclusions [12], a solution of (2.6) exists in $[t_0, \theta]$. Let $\mathrm{Sol}_0^\varepsilon(t_0, x_0, z_0, s_+)$ be the set of all such solutions $(x_0^\varepsilon(\cdot), z_0^\varepsilon(\cdot))$, and let $\mathrm{Sol}^\varepsilon(t_0, x_0, z_0, s_+)$ be the set of all solutions $(x^\varepsilon(\cdot), z^\varepsilon(\cdot))$ of the differential inclusion

$$(\dot{x}_\varepsilon(t), \dot{z}_\varepsilon(t)) \in \operatorname{co}\operatorname{pr}_{x,z} M_+^\varepsilon(t, x_\varepsilon(t), Y_+^\varepsilon(t, x_\varepsilon(t), s_+), s_+)$$
$$x_\varepsilon(t_0) = x_0, \quad z_\varepsilon(t_0) = z_0 \tag{2.7}$$

Obviously

$$\mathrm{Sol}_0^\varepsilon(t_0, x_0, z_0, s_+) \subset \mathrm{Sol}^\varepsilon(t_0, x_0, z_0, s_+) \tag{2.8}$$

For the selected trajectory $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \varepsilon - \mathrm{Sol}\ (t_0, x_0, z_0, s_+)$ let us estimate the difference between $(x^\varepsilon(t), z^\varepsilon(t))$ and $(x_\varepsilon(t), z_\varepsilon(t))$—the point on the trajectory $x_\varepsilon(\cdot), z_\varepsilon(\cdot)) \in \mathrm{Sol}^\varepsilon\ (t_0, x_0, z_0, s_+)$ closest to $(x^\varepsilon(\cdot), z^\varepsilon(\cdot))$, where $t \in [t_0, \theta]$.

By construction, this distance does not exceed the distance between $(x^\varepsilon(t), z^\varepsilon(t))$ and $(x^\varepsilon{}_0(t), z^\varepsilon{}_0(t))$—the point on the trajectory $(x_0^\varepsilon(\cdot), z_0^\varepsilon(\cdot)) \in \mathrm{Sol}_0^\varepsilon(t_0, x_0, z_0, s_+)$, closest to $(x^\varepsilon(\cdot), z^\varepsilon(\cdot))$.

Using Proposition 1, we obtain the estimate

$$\| x^\varepsilon(t) - x_\varepsilon(t) \| \leq \| x^\varepsilon(t) - x_0^\varepsilon(t) \| \leq \int_{t_0}^{t} \| \dot{x}^\varepsilon(\tau) - \dot{x}_0^\varepsilon(\tau) \| d\tau \leq \int_{t_0}^{t} \operatorname{dist}(\operatorname{pr}_x M_+^\varepsilon(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), s_+)$$
$$\operatorname{pr}_x M_+^\varepsilon(\tau, x_0^\varepsilon(\tau), Y_0^\varepsilon(\tau, x_0^\varepsilon(\tau), s_+), s_+))d\tau \leq \int_{t_0}^{t} \operatorname{dist}(\operatorname{pr}_x M_+^\varepsilon(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), s_+),$$
$$\operatorname{pr}_x M_+^\varepsilon(\tau, x_0^\varepsilon(\tau), y_0^\varepsilon(\tau), s_+))d\tau, \tag{2.9}$$

$$\| z^\varepsilon(t) - z_\varepsilon(t) \| \leq \| z^\varepsilon(t) - z_0^\varepsilon(t) \| \leq \int_{t_0}^{t} \| \dot{z}^\varepsilon(\tau) - \dot{z}_0^\varepsilon(\tau) \| d\tau \leq \int_{t_0}^{t} \operatorname{dist}(\operatorname{pr}_z M_+^\varepsilon(\tau, x^\varepsilon(\tau), y^\varepsilon(\tau), s_+)$$
$$\operatorname{pr}_z M_+^\varepsilon(\tau, x_0^\varepsilon(\tau), y_0^\varepsilon(\tau), s_+))d\tau \tag{2.10}$$

where $y_0^\varepsilon(\cdot):[t_0, \theta] \mapsto y_0^\varepsilon(t) \in Y_0^\varepsilon(t, x^\varepsilon{}_0(t), s_+)$ is some measurable function, and by (2.4)

$$\| y^\varepsilon(t) - y_0^\varepsilon(t) \| = \operatorname{dist}(y^\varepsilon(t), Y_+^\varepsilon(t, x_0^\varepsilon(t), s_+))$$

Taking conditions A.1a, A.1c and the properties of the dist operation into consideration, we continue estimate (2.9)

$$\| x^\varepsilon(t) - x_0^\varepsilon(t) \| \leq \int_{t_0}^{t} L^\varepsilon\{\| x^\varepsilon(\tau) - x_0^\varepsilon(\tau) \| + \| y^\varepsilon(\tau) - y_0^\varepsilon(\tau) \|\} d\tau \leq$$
$$\leq \int_{t_0}^{t} L^\varepsilon\{\| x^\varepsilon(\tau) - x_0^\varepsilon(\tau) \| + \operatorname{dist}(y^\varepsilon(\tau), Y_+^\varepsilon(\tau, x^\varepsilon(\tau), s_+)) +$$
$$+ \operatorname{dist}(Y_+^\varepsilon(\tau, x^\varepsilon(\tau), s_+), Y_+^\varepsilon(\tau, x_0^\varepsilon(\tau), s_+)\} d\tau \leq \int_{t_0}^{t} L^\varepsilon\{(1+\nu)$$
$$\| x^\varepsilon(\tau) - x_0^\varepsilon(\tau) \| + \operatorname{dist}(y^\varepsilon(\tau), Y_+^\varepsilon(\tau, x^\varepsilon(\tau), s_+))\} d\tau \tag{2.11}$$

By condition A.1d, relationships (1.10) will hold for the fast variable $y^\varepsilon(\cdot)$ of the selected solution $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \varepsilon - \mathrm{Sol}\ (t_0, x_0, z_0, s_+)$, that is

$$\operatorname{dist}(y^\varepsilon(t), Y_+^\varepsilon(t, x^\varepsilon(t), s_+)) \leq \operatorname{diam} D_0 = d_0 \quad \text{for } t \geq t_0 \tag{2.12}$$

$$y^\varepsilon(t) \in Y_+^\varepsilon(t, x^\varepsilon(t), s_+)) \quad \text{for } t \in [t_0 + \delta(\varepsilon), \theta] \tag{2.13}$$

Using estimates of the Gronwall type, we deduce from (2.11) and (2.12) that for $t \in [t_0, t_0 + \delta(\varepsilon)]$

$$\| x^\varepsilon(t) - x_0^\varepsilon(t) \| \leq \frac{d_0}{L^\varepsilon}(e^{L^\varepsilon(1+v^\varepsilon)\delta(\varepsilon)} - 1) = \varphi(\varepsilon) \tag{2.14}$$

where $\varphi(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. From (2.11), (2.13) and (2.14) we deduce that for $t \in [t_0 + \delta(\varepsilon), \theta]$

$$\| x^\varepsilon(t) - x_0^\varepsilon(t) \| \leq \| x^\varepsilon(t_0 + \delta(\varepsilon)) - x_0^\varepsilon(t_0 + \delta(\varepsilon)) \| + \int_{t_0+\delta(\varepsilon)}^{\theta} L^\varepsilon(1+v^\varepsilon) \| x^\varepsilon(\tau) - x_0^\varepsilon(\tau) \| d\tau \tag{2.15}$$

$$\begin{aligned} \| x^\varepsilon(t) - x_0^\varepsilon(t) \| &\leq \exp[L^\varepsilon(1+v^\varepsilon)(t - t_0 - \delta(\varepsilon))] \| x^\varepsilon(t_0 + \delta(\varepsilon)) - x_0^\varepsilon(t_0 + \delta(\varepsilon)) \| \leq \\ &\leq \exp[L^\varepsilon(1+v^\varepsilon)(\theta - t_0)]\varphi(\varepsilon) = \rho(\varepsilon) \end{aligned} \tag{2.16}$$

where $\rho(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. A similar estimate is obtained for $\| z^\varepsilon(t) - z_0^\varepsilon(t) \|$.

Summing up these arguments and, in particular, the remark that the above discussions apply both for the upper and lower characteristic complexes occurring in condition A.1, we conclude that the following proposition holds.

Lemma 1. For any compact sets D and D_0 as in condition A.1d, mappings $(0, \varepsilon_*] \mapsto R_+ \times R_+ : \varepsilon \mapsto (\alpha(\varepsilon), \rho(\varepsilon))$ exist such that $\alpha(\varepsilon) \downarrow 0$, $\rho(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, and for any $(t_0, x_0) \in D$, $y_0 \in D_0$, $z_0 \in R$, $s' = s_+ \in S_+$ ($s' = s_- \in S_-$), $\varepsilon \in (0, \varepsilon_*]$, $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \varepsilon - \mathrm{Sol}\,(t_0, x_0, y_0, z_0, s')$ a $(x^\varepsilon(\cdot), z^\varepsilon(\cdot)) \in \mathrm{Sol}^\varepsilon\,(t_0, x_0, z_0, s')$ exists such that

$$\| x^\varepsilon(\tau) - x_\varepsilon(\tau) \| \leq \rho(\varepsilon), \quad \| z^\varepsilon(\tau) - z_\varepsilon(\tau) \| \leq \rho(\varepsilon) \tag{2.17}$$

$$\text{for } \tau \in [t_0 + \delta(\varepsilon), \theta]$$

for $\tau \in [t_0 + \delta(\varepsilon), \theta]$.

Remark 1. Let $G_\varepsilon(t_0, x_0, y_0, z_0, s')$ denote the attainable domain of system (2.1) at time τ, and let $G_\varepsilon^{\rho(\varepsilon)}(t_0, x_0, z_0, s')$ be the closed $\rho(\varepsilon)$-neighbourhood of the attainable domain of system (2.7). Then condition (2.17) may be rewritten as

$$\mathrm{pr}_{x,z}\, G_\varepsilon(t_0, \tau, x_0, y_0, z_0, s') \subset G_\varepsilon^{\rho(\varepsilon)}(t_0, \tau, x_0, z_0, s') \tag{2.18}$$

where $\rho(\varepsilon)$, the quantity defined in (2.16), is the same for all $(t_0, x_0) \in D$, $y_0, \in D_0$, $z_0 \in R$.

Define functions

$$\begin{aligned} v_\varepsilon^0(t, x) &= \inf_{s_+ \in S_+} \min_{y \in Y_+^\varepsilon(t, x, s_+)} v^\varepsilon(t, x, y) \\ w_\varepsilon^0(t, x) &= \sup_{s_- \in S_-} \max_{y \in Y_-^\varepsilon(t, x, s_-)} w^\varepsilon(t, x, y) \end{aligned} \tag{2.19}$$

where $v^\varepsilon(t, x, y)$ is an upper minimax solution of the singularly perturbed problem $\mathbf{P}^\varepsilon$ and $w^\varepsilon(t, x, y)$ is a lower minimax solution of problem $\mathbf{P}^\varepsilon$. Note that previously [1] the analogous operations min and max were considered on sets of attraction Y that did not depend on t, x, s_+ and s_-. Constructions like (2.19) are also used in [13]. By the theory of minimax solutions [9, 11], for any $(t, x, y) \in G$, $s_+ \in S_+$, $s_- \in S_-$ we have a non-empty intersection

$$M_+^\varepsilon(t, x, s_+) \cap M_-^\varepsilon(t, x, s_-) \neq 0$$

Using this property and the assumption (A.1*c) that the sets of attraction are Lipschitz-continuous, it is not difficult to show, by *reductio ad absurdum*, that for any $(t, x, y) \in G$, $s_+ \in S_+$, $s_- \in S_-$ the corresponding sets of attractions from A.1* satisfy the condition

$$Y_+^\varepsilon(t, x, s_+) \cap Y_-^\varepsilon(t, x, s_-) \neq 0 \tag{2.20}$$

If we now set

$$v^{\varepsilon}(t,x,y)=w^{\varepsilon}(t,x,y)=u^{\varepsilon}(t,x,y)$$

in (2.19), where $u^{\varepsilon}(t,x,y)$ is a minimax of problem $\mathbf{P}^{\varepsilon}$, we can deduce from (2.20) that for any (t,x)

$$v_{\varepsilon}^{0}(t,x)\leqslant w_{\varepsilon}^{0}(t,x) \tag{2.21}$$

Lemma 2. For any $(t_0,x_0)\in D, y_0\in D_0, s_+\in S_+$ $(s_-\in S_-), z_0=z_0^+\geqslant v_{\varepsilon}^0(t_0,x_0)$ $(z_0=z_0^-\leqslant w_{\varepsilon}^0(t_0,x_0))$, $\tau\in[t_0+\delta(\varepsilon),\theta]$ points $(x_+^*,z_+^*)\in G_{\varepsilon}^{\rho(\varepsilon)}(t_0,\tau,x_0,z_0^+,s_+)$ and $(x_-^*,z_-^*)\in G_{\varepsilon}^{\rho(\varepsilon)}(t_0,\tau,x_0,z_0^-,s_-)$ exist such that

$$(\tau,x_+^*,z_+^*)\in \operatorname{epi} v_{\varepsilon}^0 \quad (\tau,x_-^*,z_-^*)\in \operatorname{hypo} w_{\varepsilon}^0$$

The proof follows the same lines as that of Proposition 1 in [1].

Using the continuity of the initial data of $\mathbf{P}^{\varepsilon}$ with respect to ε, we see that for all $(t,x)\in[0,\theta]\times R^n$, $s_+\in S_+, s_-\in S_-$ the following convergence relations hold in the Hausdorff metric as $\varepsilon\downarrow 0$

$$Y_{\pm}^{\varepsilon}(t,x,s_+)\mapsto Y_{\pm}^{0}(t,x,s_+)$$
$$\operatorname{co}\operatorname{pr}_{x,z} M_{\pm}^{\varepsilon}(t,x,Y_{\pm}^{\varepsilon}(t,x,s_{\pm}),s_{\pm})\mapsto M_{\pm}^{0}(t,x,s_{\pm})$$

It follows from assumptions A.1a and A.1c that for any $s_{\pm}\in S_{\pm}$ the multi-valued mappings $(t,x)\mapsto M_{\pm}^0(t,x,s_{\pm})$ are convex- and compact-valued and satisfy a Lipschitz condition with constant $L^0=\lim_{\varepsilon\downarrow 0}L^{\varepsilon}(1+\nu^{\varepsilon})$. It follows from condition A.2 that the complexes (S_+,M_+^0), (S_-,M_-^0) are upper and lower characteristic complexes, respectively, in an unperturbed Cauchy problem $\mathbf{P}$, where the Hamiltonian may be represented in the form

$$H(t,x,s)=\max_{s_+\in S_+}\min\{\langle f,s\rangle-g:(f,g)\in M_+^0(t,x,s_+)\}=$$
$$=\min_{s_-\in S_-}\max\{\langle f,s\rangle-g:(f,g)\in M_-^0(t,x,s_-)\} \tag{2.22}$$

Modifying the scheme of the proof of Proposition 2 in [1], taking into account the non-stationary nature of the sets of attraction, we deduce the following fact.

Lemma 3. The function

$$v^{\#}(t,x)=\liminf_{\varepsilon\downarrow 0,(t',x')\mapsto(t,x)} v_{\varepsilon}^{0}(t',x') \tag{2.23}$$

is an upper minimax solution of problem $\mathbf{P}$, and the function

$$w^{\#}(t,x)=\limsup_{\varepsilon\downarrow 0,(t',x')\mapsto(t,x)} w_{\varepsilon}^{0}(t',x') \tag{2.24}$$

is a lower minimax solution of problem $\mathbf{P}$ (1.14).

It follows from the properties of upper and lower minimax solutions of problem $\mathbf{P}$ that for all (t,x)

$$v^{\#}(t,x)\geqslant w^{\#}(t,x)$$

and it follows from condition (2.21) and constructions (2.23) and (2.24) that

$$v^{\#}(t,x)\leqslant w^{\#}(t,x)$$

Thus, for all $(t,x,y)\in D\times D_0$

$$v^{\#}(t,x)=w^{\#}(t,x)=u(t,x)=\lim_{\varepsilon\downarrow 0}u^{\varepsilon}(t,x,y) \tag{2.25}$$

It follows from the estimates obtained in Lemma 1 that the convergence in (2.25) is uniform in any chosen compact sets $D\subset[0,\theta]\times R^n$ and $D_0\subset R^l$. This completes the proof of Theorem 1.

Remark 2. Using conditions A.1, (1.13) and B.5, we can derive the following representation for the Hamiltonian $H(t,x,s)$ in the unperturbed problem $\mathbf{P}$

$$H(t,x,s)=\max_{s_+\in S_+}\min_{y^*\in Y^0(t,x,s_+)} H^0(t,x,y^*,s,0)=\min_{s_-\in S_-}\max_{y_*\in Y^0(t,x,s_-)} H^0(t,x,y_*,s,0) \qquad (2.26)$$

where $H^0(t,x,y,s,0) = \lim_{\varepsilon\downarrow 0} H^\varepsilon(t,x,y,s,0)$.

Remark 3. The proof of Theorem 2 reduces to verifying that the role of the admissible complexes in condition A.1 of Theorem 1 may be played by sets $S_+ = S_- = \{(p,q) \in R^n \times R^l\}$ and multi-valued mappings of the form

$$\begin{aligned}
&(t,x,y)\mapsto M_+^\varepsilon(t,x,y,p,q)=\{(f,g,r)\in R^n\times R^l\times R:\\
&\|f\|\leqslant\lambda^\varepsilon(x,y),\quad g\in k^\varepsilon(t,x,y)+F_+^\varepsilon(t,x,q),\\
&r=\langle f,p\rangle-H^\varepsilon(t,x,y,p,0)\}\\
&(t,x,y)\mapsto M_-^\varepsilon(t,x,y,p,q)=\{(f,g,r)\in R^n\times R^l\times R:\\
&\|f\|\leqslant\lambda^\varepsilon(x,y),\quad g\in k^\varepsilon(t,x,y)+F_-^\varepsilon(t,x,q),\\
&r=\langle f,p\rangle-H^\varepsilon(t,x,y,p,0)\}
\end{aligned}$$

Condition A.1*d implies an exponential estimate for the rate at which the "fast" components of the generalized characteristics approach the corresponding sets of attraction, so that condition A.1d holds.

3. EXAMPLES

Conditions A.1, A.2, B.1–B.5 are satisfied, for example, in the model examples if [1].

For the first example, the upper characteristic complexes and corresponding sets of attraction satisfying condition A.1 are as follows:

$$\begin{aligned}
&s_+=q',\quad S_+=Q\\
&M_+^\varepsilon(t,x,y_1,y_2,q')=\mathrm{co}\{(f(t,x,y_1,y_2),h_1^\varepsilon(y_1,p'),h_2^\varepsilon(y_2,q'),\\
&g(t,x,y_1,y_2)):\mathbf{p}'\in P\}\\
&h_1^\varepsilon(y_1,\mathbf{p}')=\frac{1}{\varepsilon}(\mathbf{p}'-y_1),\quad h_2^\varepsilon(y_2,q')=\frac{1}{\varepsilon}(q'-y_2)\\
&Y_+^\varepsilon(t,x,q')=P^\varepsilon\times Q^\varepsilon\\
&M_+^0(t,x,q')=\mathrm{co}\{(f(t,x,\mathbf{p}',q'),g(t,x,\mathbf{p}',q')):\mathbf{p}'\in P\}\\
&Y_+^0(t,x,q')=P\times Q
\end{aligned}$$

where P^ε and Q^ε are closed ε-neighbourhoods of the sets P and Q.

In order to construct suitable lower characteristic complexes, we need only interchange the roles of p' and q' and P and Q in the constructions, leaving the sets of attraction as before.

For the second example, the upper characteristic complexes and corresponding sets of attraction satisfying condition A.1d are as follows:

$$\begin{aligned}
&s_+=\beta,\quad S_+=B\\
&M_+^\varepsilon(t,x,y,\beta)=\mathrm{co}\{(f(t,x,y),\frac{1}{\varepsilon}(\xi-y),g(t,x,y)):\xi\in Y(t,x,\beta)\}\\
&M_+^0(t,x,\beta)=\mathrm{co}\{(f(t,x,\xi),g(t,x,\xi)):\xi\in Y(t,x,\beta)\}\\
&Y_+^\varepsilon(t,x,\beta)=Y(t,x,\beta)^\varepsilon,\quad Y_+^0(t,x,\beta)=Y(t,x,\beta)
\end{aligned}$$

where $Y(t,x,\beta)^\varepsilon$ is the closed ε-neighbourhood of the set $Y(t,x,\beta)$.

To construct suitable lower characteristic complexes and sets of attraction, we need only interchange the roles of β and α and B and A, in the constructions.

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